Cox and Munk's Sea Surface Slope Variance

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The data analysis portion of Cox and Munk's classic sun glitter experiment is reviewed, and their extrapolation of the experimental data is criticized.

Cox and Munk's [1954, 1956] expressions for the sea surface slope variance are frequently cited in the literature. This paper reviews the basis of these expressions and emphasizes that they are rough approximations based on a rather arbitrary data extrapolation. However, an accurate lower bound to the variance can be deduced from the data.

From photometric photographs of sun glitter on the sea surface, Cox and Munk calculate the probability density function (pdf) for the crosswind and upwind surface slopes, Z_c and Z_u. They initially assume that the slope pdf is approximately Gaussian and can be represented by

$$P(m, \alpha) = [1 + T(m, \alpha)](2\pi\sigma_c\sigma_u)^{-1} \exp \{-(m/2\sigma_c\sigma_u)^2[\sigma_c^2 + \sigma_u^2 + (\sigma_c^2 - \sigma_u^2) \cos 2\alpha]\} \quad (1)$$

where $m = (Z_c^2 + Z_u^2)^{1/2}$, $\alpha = \tan^{-1}(Z_c/Z_u)$, and σ_c^2 and σ_u^2 are the crosswind and upwind slope variances. The function $T(m, \alpha)$ accounts for deviations from a Gaussian pdf. Requiring that $|T(m, \alpha)| < 1$, they show that the logarithm of (1) takes the form

$$\ln [P(m, \alpha)] = \sum_{j=0}^{4} \sum_{k=0}^{\langle 2-(1/2)j \rangle} -a_{jk} m^{j+2k} \cos j\alpha \quad (2)$$

where the angle brackets denote truncation. The harmonics higher than cos 4α are assumed negligible. The film density at various points on the photographs is measured, and at each point the slope required to reflect the sun's rays into the camera is calculated, thereby relating film density to the logarithm of slope probability. They fit these slope probability data to the analytical form specified by (2) and find all the a_{jk} coefficients except for the normalization constant a_{00}. Their resulting empirical expression for the unnormalized slope pdf will be denoted by

$$P^*(m, \alpha) = P(m, \alpha) \exp (a_{00}) \quad (3)$$

In theory, the sea slope variances are given by

$$\sigma_i^2 = \left[\int_0^\infty dm \int_{-\pi}^{\pi} d\alpha\, m^3 \Psi_i P^*(m, \alpha)\right] \cdot \left[\int_0^\infty dm \int_{-\pi}^{\pi} d\alpha\, m P^*(m, \alpha)\right]^{-1} \quad (4)$$

where $i = c$ or u, $\Psi_c = \sin^2\alpha$, and $\Psi_u = \cos^2\alpha$. Cox and Munk find that (4) can be conveniently rewritten in terms of the 'incomplete variance function' $F_i(U)$, which is insensitive to variations in wind speed and which can be expressed in a convergent power series in U.

$$F_i(U) = \left[\int_0^U du \int_{-\pi}^{\pi} d\alpha\, u\Psi_i P^*(m, \alpha)\right] \cdot \left[\int_0^U du \int_{-\pi}^{\pi} d\alpha\, P^*(m, \alpha)\right]^{-1} \quad (5)$$

where $u = a_{01}m^2$. Changing the integration variable from m to u in (4) gives

$$\sigma_i^2 = \lim_{U\to\infty} F_i(U)/a_{01} \quad (6)$$

All of the wind speed dependence is contained in the coefficient a_{01}.

In practice, the above limit cannot be evaluated from the data. Sunlight scattered from submerged particles and reflected skylight produce a background in which the sun glitter from large and infrequent slopes is lost. In general, no data are available for $a_{01}m^2 > 4$, and $F_i(U)$ is only known for $U \leq 4$. To account for the missing data, Cox and Munk extrapolate $F_i(U)$ out to $U = 8$. Their reported slope variances are then calculated from

$$(\sigma_i^2)_{\text{CM}} = F_i(8)/a_{01} \quad (7)$$

Figure 1 shows that $F_i(U)$ diverges for large U. This divergence is obviously unrealistic and shows that the extrapolation completely breaks down for very steep slopes. It cannot be determined where this breakdown begins, and the choice of $U = 8$ is arbitrary. However, one can make an accurate lower-bound estimate of the variance given by

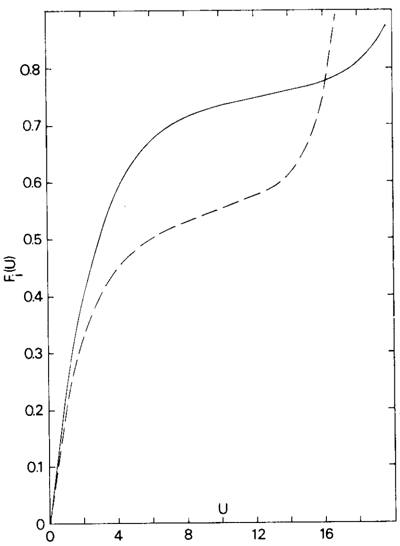

Fig. 1. Cox and Munk's incomplete variance function for the upwind (solid curve) and crosswind (dashed curve) cases.

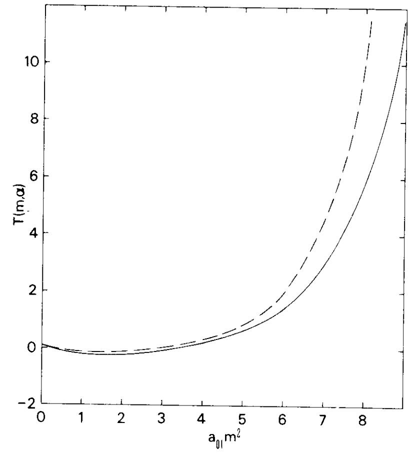

Fig. 2. Deviation of the sea slope pdf from a Gaussian distribution for which $T(m, \alpha) = 0$. The solid and dashed curves correspond to the upwind and crosswind cases.

$$(\sigma_i^2)_{LB} = F_i(4)/a_{01} \tag{8}$$

Noting that $F_i(4)/F_i(8) = 0.8$, one can calculate $(\sigma_i^2)_{LB}$ by multiplying Cox and Munk's variances by 0.8.

A second argument against the Cox and Munk extrapolation can be made. Their assumptions that $F_i(U)$ can be safely extrapolated up to $U = 8$ and that slopes for which $U > 8$ negligibly contribute to σ_i^2 are inconsistent with their requirement that $|T(m, \alpha)| < 1$. If their assumptions were correct, the normalized slope pdf $P(m, \alpha)$ could be determined from (3) by evaluating the normalization constant a_{00} given by

$$a_{00} = \ln\left[(2a_{01})^{-1}\int_0^8 du \int_{-\pi}^{\pi} d\alpha\ P^*(m, \alpha)\right] \tag{9}$$

One could then solve for $T(m, \alpha)$ by substituting $(\sigma_i^2)_{CM}$ for the variances appearing in (1). Figure 2 shows the results of such calculations. $T(m, \alpha)$ is plotted versus $a_{01}m^2$ for a wind speed of 14 m/s and for the upwind ($\alpha = 0$) and the crosswind ($\alpha = \pi/2$) cases. The region of extrapolation corresponds to $4 < a_{01}m^2 \leq 8$, and the requirement that $|T(m, \alpha)| < 1$ is violated for $a_{01}m^2 > 5$.

The lower bound to the sea slope variances given by (8) is consistent with the variances reported by other experimenters [*Wu,* 1971; *Pierson and Stacy,* 1973] and with the variances required for best agreement between microwave scattering theory and radiometric measurements [*Wentz,* 1975].

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